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LIFTING OF NICHOLS ALGEBRAS OF TYPE B_2*

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ABSTRACT

We compute liftings of the Nichols algebra of a Yetter-Drinfeld module of Cartan type B_2 subject to the small restriction that the diagonal elements of the braiding matrix are primitive nth roots of 1 with odd $n \neq 5$. As

* With an appendix "A generalization of the q-binomial theorem" with Ian Rutherford, Department of Mathematics and Computer Science, Mount Allison University, Sackville, N.B., Canada E4L 1E6.

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well, we compute the liftings of a Nichols algebra of Cartan type A_2 if the diagonal elements of the braiding matrix are cube roots of 1; this case was not completely covered in previous work of Andruskiewitsch and Schneider. We study the problem of when the liftings of a given Nichols algebra are quasi-isomorphic. The Appendix (with I. Rutherford) contains a generalization of the quantum binomial formula. This formula was used in the computation of liftings of type B_2 but is also of interest independent of these results.

1. Introduction and preliminaries

Let k be an algebraically closed field of characteristic zero. Several classification results for finite dimensional pointed Hopf algebras have been obtained in recent years (see [1] for a survey). The most powerful general method for classifying such Hopf algebras is the lifting method developed by N. Andruskiewitsch and H.-J. Schneider. If A is a finite dimensional pointed Hopf algebra with coradical $k\Gamma$, Γ a group, then there exists a Hopf algebra projection from $gr(A)$, the associated graded Hopf algebra, to $k\Gamma$, and this projection splits the inclusion of $k\Gamma$ in $gr(A)$ as the degree 0 component. Then the subalgebra R of $k\Gamma$-coinvariants of $gr(A)$, called the diagram of A, has a Hopf algebra structure in the braided category ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ of Yetter–Drinfeld modules over $k\Gamma$. One can also associate to A the Yetter–Drinfeld module V of primitive elements of R, called the infinitesimal braiding of A. The Hopf algebra $gr(A)$ can be reconstructed by bosonization from R, i.e., $gr(A) \simeq R\#k\Gamma$, the biproduct in the sense of D. Radford or S. Majid. The lifting procedure consists first in finding all the possible diagrams R, then bosonizing to $gr(A)$, and finally lifting the information (i.e., presentation by generators and relations) from $gr(A)$ to A.

Assume that Γ is a fixed finite abelian group. If V is a Yetter–Drinfeld module, the Nichols algebra $\mathcal{B}(V)$ is a graded Hopf algebra in the category ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ with $k1$ as the homogeneous component of degree 0, V as the homogeneous component of degree 1, and $\mathcal{B}(V)$ is generated in degree 1 as an algebra. Nichols algebras were introduced in [15] (see [2] for a general presentation of the construction of and recent developments in Nichols algebras). Their role in the classification theory for pointed Hopf algebras was emphasized in [5]. A fundamental question is whether the diagram R of A is just the Nichols algebra $\mathcal{B}(V)$ of the infinitesimal braiding of A. A positive answer to this question is equivalent to proving the conjecture that any finite dimensional pointed Hopf algebra is generated as an algebra by the grouplike elements and the skew-primitive elements. Up to this conjecture, the lifting method for classifying finite dimensional pointed Hopf algebras A with coradical $k\Gamma$ reduces to finding all the Yetter–Drinfeld modules V such that $\mathcal{B}(V)$ is finite dimensional, then describing the Nichols algebra $\mathcal{B}(V)$ by generators and relations for any such V, and finally finding A such that the associated graded Hopf algebra $gr(A)$ is isomorphic to the biproduct $\mathcal{B}(V)\#k\Gamma$. Such an A is called a lifting of $\mathcal{B}(V)\#k\Gamma$.

A major step in the classification problem was done in [5], where the approach was from the point of view of Lie theory. For certain Hopf algebras A (or for any A if the exponent of Γ is prime), the infinitesimal braiding has a generalized Cartan matrix as an invariant. Then the dimension of $\mathcal{B}(V)$ and the structure of this algebra, reflecting that of A, depend on this Cartan matrix and on its Dynkin diagram. As an example, the lifting method was used in [6] to describe liftings of Nichols algebras of Cartan type A_2, and as a consequence classify pointed Hopf algebras of dimension p^4, with p an odd prime. Also, the lifting method was used in [10] to classify pointed Hopf algebras of dimension 32.

The main aim of this paper is to compute liftings of Nichols algebras of Cartan type B_2. The description of these Nichols algebras is known (see [5] and [17]). We follow the general approach that was used in [6] for type A_2. The problem of lifting the generators and relations from $gr(A)$ to A has a combinatorial nature, and compared to the A_2 case, the case of Cartan type B_2 requires more complicated combinatorics. This is because the structure of the positive roots, which define a system of generators for the Nichols algebra, is more complicated in type B_2. To deal with these combinatorial difficulties, we use a generalization of the quantum binomial formula presented in the Appendix. In Section 2 we compute the liftings in type B_2. We require that the diagonal elements of the braiding matrix are primitive n-th roots of odd order not equal to 5. In fact, in type A_2 there was also a case for which the computation in [6] failed, more precisely the case where the diagonal elements of the braiding matrix were primitive roots of unity of order 3. In Section 3, we show how this remaining case can be completed.

The first examples of infinite families of nonisomorphic Hopf algebras of the same dimension were liftings of quantum linear spaces [4], [9], [8] or [7], and E. Müller's family of nonisomorphic nonpointed Hopf algebras with nonpointed duals [14]. However, A. Masuoka [12] showed that these infinite families consist of Hopf algebras that are all quasi-isomorphic, i.e., that any element of the family is a cocycle twist of any other, or equivalently, their categories of comodules are monoidally Morita–Takeuchi equivalent (see [12] or [17]). We prove in Section 3 that for $n \neq 5$ and V of type B_2 or for $n \neq 3$ and V of type A_2, any two liftings of $\mathcal{B}(V)\#k\Gamma$ are quasi-isomorphic.

Recall from any standard text (such as [11]) the notation for q-factorials and q-binomial coefficients. Set $(0)_q = 0$ and for $n > 0, (n)_q = q^{n-1} + q^{n-2} + \cdots + 1$. Set $(0)!_q = 1$ and for $n > 0, (n)!_q = (n)_q(n-1)_q \cdots 1$. Then

$$\binom{n}{i}_q = \frac{(n)!_q}{(n-i)!_q(i)!_q} \quad \text{where } 0 \leq i \leq n.$$

If i, n or $n-i$ is negative, then we set $\binom{n}{i}_q = 0$.

THEOREM 1.1: (i) *(The q-Pascal identity). For $n \geq k \geq 1$,*

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

(ii) *(The q-binomial theorem). For x, z elements of some k-algebra with $zx = qxz$, $q \in k^*$, then*

$$(x+z)^n = \sum_{i=0}^{n} \binom{n}{i}_q x^i z^{n-i}. \qquad \blacksquare$$

For A a pointed Hopf algebra with coradical $k\Gamma$, we denote by $P(A)_{g,h}$ or $P_{g,h}$, if A is clear, the set $\{x\colon x \in A, \Delta(x) = g \otimes x + x \otimes h\}$. For $\chi \in \hat{\Gamma}$, $P^{\chi}_{g,h} = \{x \in P_{g,h} : lxl^{-1} = \chi(l)x \text{ for all } l \in \Gamma\}$.

Notation: We write P_g for $P_{g,1}$ and P^{χ}_g for $P^{\chi}_{g,1}$.

For any coalgebra C, $gr(C)$, the graded vector space $C_0 \oplus C_1/C_0 \oplus C_2/C_1 \oplus \cdots$ is a graded coalgebra. If A is a pointed Hopf algebra, then $gr(A)$ is a graded Hopf algebra.

A Yetter–Drinfeld module $V \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ is a vector space with a left action of $k\Gamma$ and a left coaction δ: $V \to k\Gamma \otimes V$, $\delta(v) = \sum v_{-1} \otimes v_0$ such that $\delta(hv) = \sum hv_{-1}h^{-1} \otimes hv_0$ for any $h \in \Gamma$ and $v \in V$.

Throughout, Γ will be a fixed finite abelian group and k an algebraically closed field of characteristic zero.

2. Liftings of Nichols algebras of type B_2

For Γ our fixed finite abelian group, let V be a Yetter–Drinfeld module over $k\Gamma$ of dimension 2. Then V has a basis $\{x_1, x_2\}$ over k such that for $i = 1, 2$, the coaction is given by $\delta(x_i) = g_i \otimes x_i$ where $g_i \in \Gamma$, and the action is given by $g \to x_i = \chi_i(g)x_i$ for some $\chi_i \in \hat{\Gamma}$, all $g \in \Gamma$. The braiding matrix of V is

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where $b_{ij} = \chi_j(g_i)$. We assume that V has Cartan type B_2, i.e., $b_{ij}b_{ji} = b_{ii}^{a_{ij}}$ for any i, j, where the a_{ij}'s are the entries of the Cartan matrix of type B_2

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Thus we have

$$b_{12}b_{21} = b_{11}^{-1}, \quad b_{21}b_{12} = b_{22}^{-2},$$

so

$$b_{12}b_{21}b_{11} = 1, \quad b_{21}b_{12}b_{22}^2 = 1, \quad b_{11} = b_{22}^2. \tag{1}$$

We assume that $b_{22} = q$ is a primitive root of unity of odd order n and therefore so is $b_{11} = q^2$. We fix the Yetter–Drinfeld module V of type B_2 as above throughout all this section.

The Nichols algebra $\mathcal{B}(V)$ has dimension n^4 (see [5] and [17]). It is presented by generators x_1, x_2, z, u subject to the relations

$$\begin{aligned} z =& x_2x_1 - b_{21}x_1x_2; \\ u =& x_2z - b_{21}b_{22}zx_2; \\ x_1z =& b_{12}zx_1; \\ x_2u =& b_{21}b_{22}^2ux_2; \\ uz =& b_{11}b_{21}zu; \\ x_1u =& b_{21}^{-1}b_{12}ux_1 + b_{21}^{-1}(b_{22}^{-1} - 1)z^2; \\ x_1^n = x_2^n =& z^n = u^n = 0. \end{aligned}$$

Then $H = \mathcal{B}(V)\#k\Gamma$ has dimension $n^4|\Gamma|$ where $|\Gamma|$ is the order of Γ. Our goal in this section is to find all Hopf algebras A such that $gr(A) = H$.

Let A be such a lifting. By [4, Lemma 5.4], we have that A has coradical $k[\Gamma]$ and $P(A)_h = k(h-1)$ unless $h = g_i, i = 1, 2$. If $h = g_i$ then P_{g_i} has dimension 2 and $P_{g_i} = k(g_i - 1) \oplus ka_i$ where $ka_i = P_{g_i}^{\chi_i}$. The image of a_i in A_1/A_0, is just x_i. Thus we have $\Delta(a_i) = g_i \otimes a_i + a_i \otimes 1$. For $h \in \Gamma$, $ha_i = \chi_i(h)a_ih$ and, in particular, $g_ja_i = \chi_i(g_j)a_ig_j = b_{ji}a_ig_j$.

Let $\rightharpoonup$ denote the adjoint action and define elements

$$c = a_2 \rightharpoonup a_1 = a_2a_1 - b_{21}a_1a_2 \quad \text{and} \quad d = a_2 \rightharpoonup c = a_2c - b_{21}b_{22}ca_2. \tag{2}$$

The following lemma will be useful in determining the multiplication of the elements a_1, a_2, c, d.

LEMMA 2.1: *Let n, χ_i, g_i be as above. The following assertions hold.*

1. *For $i = 1, 2$, if $\chi_i^n \neq \epsilon$, then $P(A)_{g_i^n}^{\chi_i^n} = 0$.*
2. *Either $(\chi_1\chi_2)^n = \epsilon$ or $P(A)_{(g_1g_2)^n}^{(\chi_1\chi_2)^n} = 0$.*
3. *Either $(\chi_1\chi_2^2)^n = \epsilon$ or $P(A)_{(g_1g_2^2)^n}^{(\chi_1\chi_2^2)^n} = 0$.*
4. *For n different from 5, $P(A)_{g_1^2g_2}^{\chi_1^2\chi_2} = 0$.*
5. *If n is different from 3 and 5, then $P(A)_{g_1g_2^3}^{\chi_1\chi_2^3} = 0$.*

Proof:

1. If $\chi_i^n \neq \epsilon$, then $P(A)_{g_i^n}^{\chi_i^n} \neq 0$ implies that $\chi_i^n = \chi_j$ and $g_i^n = g_j$ where $j = 1$ or 2. If $\chi_i^n = \chi_i$ then $b_{ii}^{n-1} = 1$, which is impossible. If $\chi_i^n = \chi_j, j \neq i$, then $1 = b_{ij}$ and $b_{ji}^n = b_{jj}$. But since $(b_{12}b_{21})^n = 1$, this is impossible.
2. Suppose $(\chi_1\chi_2)^n \neq \epsilon$ and $P(A)_{(g_1g_2)^n}^{(\chi_1\chi_2)^n} \neq 0$. Then $(\chi_1\chi_2)^n = \chi_i$, so that $b_{ii} = b_{ij}^n$. Also $(g_1g_2)^n = g_i$, so that $b_{ii} = b_{ji}^n$. This contradicts $(b_{ij}b_{ji})^n = 1$.
3. Suppose $(\chi_1\chi_2^2)^n \neq \epsilon$ and $P(A)_{(g_1g_2^2)^n}^{(\chi_1\chi_2^2)^n} \neq 0$. If $(\chi_1\chi_2^2)^n = \chi_1$ and $(g_1g_2^2)^n = g_1$, then $b_{11}^{n-1} = b_{12}^{-2n} = b_{21}^{-2n}$ so that $1 = b_{11}^2$, which is impossible. If $(\chi_1\chi_2^2)^n = \chi_2$ and $(g_1g_2^2)^n = g_2$, then $b_{12}^{2n-1} = 1 = b_{21}^{2n-1}$. But then $1 = (b_{12}b_{21})^{2n-1} = (b_{12}b_{21})^{-1} = b_{11}$, also a contradiction.
4. For $n \neq 5, \chi_1^2\chi_2 \neq \epsilon$. For if $\chi_1^2\chi_2 = \epsilon$, then $b_{11}^2b_{12} = 1 = b_{21}^2b_{22}$. But then $b_{12} = q^{-4}$ and $b_{21}^2 = q^{-1}$ so that $q^{-8}q^{-1} = (b_{12}b_{21})^2 = q^{-4}$ so that $q^5 = 1$, which would imply that $n = 5$. Thus if $P(A)_{g_1^2g_2}^{\chi_1^2\chi_2} \neq 0$, then $g_1^2g_2 = g_i$ for $i = 1, 2$. But if $g_1^2g_2 = g_1$, then $g_1g_2 = 1$ so that $b_{11}b_{21} = 1$ and $b_{12}b_{22} = 1$. The first equation implies $b_{12} = 1$ so that the second implies $b_{22} = q = 1$, which is impossible. If $g_1^2g_2 = g_2$ then $g_1^2 = 1$ so that $q^4 = 1$, which contradicts the assumption that n is odd.
5. If $\chi_1\chi_2^3 = \epsilon$ then $b_{11}b_{12}^3 = 1 = b_{21}b_{22}^3$. Then $b_{12}^3 = q^{-2}$ and $b_{21} = q^{-3}$. Thus $(b_{12}b_{21})^3 = q^{-11}$ but $(b_{12}b_{21})^3 = q^{-6}$ by (1). Since $n \neq 5$, this is a contradiction. Thus if $P(A)_{g_1g_2^3}^{\chi_1\chi_2^3} \neq 0$, $P(A)_{g_1g_2^3}^{\chi_1\chi_2^3} = P(A)_{g_i}^{\chi_i}$ for $i = 1, 2$. If $g_1g_2^3 = g_1$, then $g_2^3 = 1$ and $b_{22}^3 = q^3 = 1$, which contradicts the fact that n is not 3. If $g_1g_2^3 = g_2$, then $g_1g_2^2 = 1$ so that $b_{11}b_{21}^2 = 1 = b_{12}b_{22}^2$. Multiplying the second equation by b_{21} yields $b_{21} = 1$ and then the first equation implies $b_{11} = 1$. But then $q = 1$, which is impossible. ∎

Straightforward calculation shows that

$$\Delta(c) = g_1g_2 \otimes c + c \otimes 1 + (1 - q^{-2})a_2g_1 \otimes a_1. \tag{3}$$

Now we compute

$$\begin{aligned}\triangle(a_1c) =&(a_1\otimes 1+g_1\otimes a_1)(g_1g_2\otimes c+c\otimes 1+(1-q^{-2})a_2g_1\otimes a_1)\\ =&a_1g_1g_2\otimes c+a_1c\otimes 1+(1-q^{-2})a_1a_2g_1\otimes a_1+g_1^2g_2\otimes a_1c\\ &+g_1c\otimes a_1+(1-q^{-2})g_1a_2g_1\otimes a_1^2\end{aligned}$$

and

$$\begin{aligned}\triangle(ca_1) =&(g_1g_2\otimes c+c\otimes 1+(1-q^{-2})a_2g_1\otimes a_1)(a_1\otimes 1+g_1\otimes a_1)\\ =&g_1g_2a_1\otimes c+ca_1\otimes 1+(1-q^{-2})a_2g_1a_1\otimes a_1\\ &+g_1^2g_2\otimes ca_1+cg_1\otimes a_1+(1-q^{-2})a_2g_1^2\otimes a_1^2.\end{aligned}$$

Then using the relations (1), we see that $a_1c-b_{12}ca_1\in P(A)^{\chi_1^2\chi_2}_{g_1^2g_2}$ and thus by Lemma 2.1 (4), if $n\neq 5$,

$$a_1c-b_{12}ca_1=0. \tag{4}$$

Similarly, using the definition of d and the comultiplication of c and a_2, we compute

$$\triangle(d)=d\otimes 1+g_1g_2^2\otimes d+q(1-q^{-2})a_2g_1g_2\otimes c+(1-q^{-2})(1-q^{-1})a_2^2g_1\otimes a_1. \tag{5}$$

Further computation shows that $da_2-b_{12}a_2d$ is $(g_1g_2^3,1)$-primitive and by Lemma 2.1 (5), if $n\neq 3,5$, then

$$da_2-b_{12}a_2d=0. \tag{6}$$

Now, using (4) and (6), we compute

$$\begin{aligned}da_1 =&(a_2c-b_{21}b_{22}ca_2)a_1\\ =&a_2(b_{21}b_{22}^2a_1c)-b_{21}b_{22}c(b_{21}a_1a_2+c)\\ =&b_{21}b_{22}^2(b_{21}a_1a_2+c)c-b_{21}^2b_{22}(b_{12}^{-1}a_1c)a_2-b_{21}b_{22}c^2\\ =&b_{21}^2b_{22}^2a_1(a_2c-b_{21}b_{22}ca_2)+b_{21}b_{22}(b_{22}-1)c^2,\end{aligned}$$

so that

$$da_1=(b_{21}b_{22})^2a_1d+(b_{21}b_{22})(q-1)c^2. \tag{7}$$

A similar computation shows that

$$cd=b_{12}dc. \tag{8}$$

Now since a_1 is $(g_1, 1)$-primitive and $q^2 = \chi_1(g_1)$ is a primitive n-th root of unity, we have that $a_1^n \in P(A)_{g_1^n}^{\chi_1^n}$. If $\chi_1^n \neq \epsilon$ then we must have that $a_1^n = 0$ since $P(A)_{g_1^n}^{\chi_1^n} = 0$, by Lemma 2.1 (1), but if $\chi_1^n = \epsilon$, then $a_1^n = \alpha(g_1^n - 1)$ for some $\alpha \in k$. By a similar argument for a_2^n and rescaling the a_i if necessary, we have

$$(9) \qquad a_i^n = \mu_i(g_i^n - 1), \quad \text{where } \mu_i \in \{0,1\}, \mu_i = 0 \text{ if } g_i^n = 1 \text{ or } \chi_i^n \neq \epsilon.$$

Remark 2.2: If $b_{21}^n \neq 1$, or equivalently $b_{12}^n \neq 1$, then $\mu_2 = 0 = \mu_1$. For suppose $b_{21}^n \neq 1$. Then $\chi_1^n(g_2) = b_{21}^n \neq 1$, so $\chi_1^n \neq \epsilon$ and then $\mu_1 = 0$. Also since $\chi_2^n(g_1) = b_{12}^n \neq 1$, we have that $\chi_2^n \neq \epsilon$, and then $\mu_2 = 0$.

Now we compute c^n and d^n; these are the most intricate computations.

By Equation (3), $\Delta(c) = X + Y + (1 - q^{-2})Z$ where $X = g_1g_2 \otimes c$, $Y = c \otimes 1$, $Z = a_2g_1 \otimes a_1$. Then, since $XY = qYX, XZ = qZX$, and q is a primitive nth root of unity, we see that $\Delta(c^n) = (X + Y + (1 - q^{-2})Z)^n = X^n + (Y + (1 - q^{-2})Z)^n$. Now

$$\begin{aligned} ZY - qYZ =& a_2g_1c \otimes a_1 - qca_2g_1 \otimes a_1 \\ =& b_{12}b_{11}a_2cg_1 \otimes a_1 - qca_2g_1 \otimes a_1 \\ =& b_{12}b_{22}^2 dg_1 \otimes a_1. \end{aligned}$$

Let T denote $dg_1 \otimes a_1$. Then it is easily checked that $ZT = q^2TZ$ and $TY = q^2YT$, so that by Remark A.7 of the Appendix, $(Y + (1 - q^{-2})Z)^n = Y^n + (1 - q^{-2})^n Z^n$. Thus

$$\begin{aligned} \Delta(c^n) =& (g_1g_2)^n \otimes c^n + c^n \otimes 1 + (1 - q^{-2})^n (a_2g_1)^n \otimes a_1^n \\ =& (g_1g_2)^n \otimes c^n + c^n \otimes 1 + (1 - q^{-2})^n b_{21}^{-n(n+1)/2} (g_1)^n \mu_2(g_2^n - 1) \otimes a_1^n \\ =& (g_1g_2)^n \otimes c^n + c^n \otimes 1 + (q^2 - 1)^n \mu_2 (g_1)^n (g_2^n - 1) \otimes a_1^n, \end{aligned}$$

since $b_{21}^n = 1$ if $\mu_2 \neq 0$ by Remark 2.2. Let

$$(10) \qquad v = c^n + \mu_2(q^2 - 1)^n a_1^n.$$

Then $\Delta(v)$ is

$$(g_1g_2)^n \otimes c^n + c^n \otimes 1 + (q^2 - 1)^n \mu_2 (g_1)^n (g_2^n - 1) \otimes a_1^n + \mu_2(q^2 - 1)^n g_1^n \otimes a_1^n + \mu_2(q^2 - 1)^n a_1^n \otimes 1.$$

It follows that $v \in P(A)_{(g_1g_2)^n}^{(\chi_1\chi_2)^n}$. Thus by Lemma 2.1 (2),

$$(11) \qquad \begin{gathered} v = c^n + \mu_2(q^2 - 1)^n a_1^n = \lambda(g_1^n g_2^n - 1) \\ \text{where } \lambda = 0 \text{ if } g_1^n g_2^n = 1 \text{ or } (\chi_1\chi_2)^n \neq \epsilon. \end{gathered}$$

Remark 2.3: If $b_{12}^n \neq 1$, then $\lambda = 0$. Indeed, $(\chi_1\chi_2)^n(g_1) = b_{12}^n \neq 1$, which forces $\lambda = 0$.

We now compute d^n. From (5), we see that $\Delta(d) = X + Y + bZ + T$ where

$$\begin{aligned} X =& g_1g_2^2 \otimes d, \\ Y =& d \otimes 1, \\ Z =& a_2g_1g_2 \otimes c \quad \text{and} \quad b = q(1-q^{-2}), \\ T =& (1-q^{-2})(1-q^{-1})a_2^2g_1 \otimes a_1. \end{aligned}$$

It is easy to check that $XY = q^2YX, ZY = q^2YZ, TY = q^2YT$, so that by Theorem (1.1)(ii), the q-binomial theorem,

$$\Delta(d^n) = (X + bZ + T)^n + Y^n$$

and it remains to compute $(X + bZ + T)^n$. Straightforward computation yields that

$$XZ = q^2ZX \quad \text{and} \quad ZT = q^2TZ$$

and we illustrate the type of calculation involved by computing XT. We have

$$\begin{aligned} XT =& (1-q^{-1})(1-q^{-2})(g_1g_2^2 \otimes d)(a_2^2g_1 \otimes a_1) \\ =& (1-q^{-1})(1-q^{-2})b_{12}^2b_{22}^4a_2^2g_1^2g_2^2 \otimes da_1 \\ =& (1-q^{-1})(1-q^{-2})b_{12}^2b_{22}^4a_2^2g_1^2g_2^2 \otimes (b_{21}^2b_{22}^2a_1d + b_{21}b_{22}(q-1)c^2) \\ =& (1-q^{-1})(1-q^{-2})(q^2a_2^2g_1^2g_2^2 \otimes a_1d + (q-1)b_{12}q^3a_2^2g_1^2g_2^2 \otimes c^2). \end{aligned}$$

Since $Z^2 = b_{12}qa_2^2g_1^2g_2^2 \otimes c^2$, we have

$$XT = q^2TX + (1-q^{-2})(1-q^{-1})(q-1)q^2Z^2. \tag{12}$$

Using Theorem A.1 in the Appendix, we see that

$$(X + bZ + T)^n = X^n + \nu(n)Z^n + T^n$$

and, by Corollary A.5, $\nu(n) = \alpha^n + \beta^n$ where $\alpha = q-1, \beta = 1-q^{-1}$ are the solutions of the equation $Y^2 - q(1-q^{-2})Y + (1-q^{-2})(1-q^{-1})(q-1)q^2/(q^2-1) = 0$. Thus $\nu(n) = (q-1)^n + (1-q^{-1})^n = 2(q-1)^n$ and we have

$$\begin{aligned} \Delta(d^n) =& (g_1g_2^2)^n \otimes d^n + d^n \otimes 1 + 2(q-1)^n(a_2g_1g_2)^n \otimes c^n \\ & + (q^2-1)^n(q-1)^n(a_2^2g_1)^n \otimes a_1^n \\ =& (g_1g_2^2)^n \otimes d^n + d^n \otimes 1 + 2(q-1)^n a_2^n g_1^n g_2^n \otimes c^n \\ & + (q^2-1)^n(q-1)^n a_2^{2n} g_1^n \otimes a_1^n, \end{aligned}$$

where $(a_2g_1g_2)^n = (b_{22}b_{12})^{n(n-1)/2}a_2^n g_1^n g_2^n$ and $b_{12}^{n(n-1)/2} = 1$ if $a_2^n \neq 0$ by Remark 2.2. Similarly $(a_2^2 g_1)^n = a_2^{2n} g_1^n$. Let

$$\omega = d^n + 2(q-1)^n \mu_2 c^n + (q^2-1)^n (q-1)^n \mu_2^2 a_1^n \tag{13}$$

and then

$$\begin{aligned}
\Delta(\omega) =&(g_1g_2^2)^n \otimes d^n + d^n \otimes 1 + 2(q-1)^n \mu_2 (g_2^n - 1)(g_1g_2)^n \otimes c^n \\
&+ (q^2-1)^n(q-1)^n \mu_2^2 (g_2^n - 1)^2 g_1^n \otimes a_1^n \\
&+ 2(q-1)^n \mu_2 g_1^n g_2^n \otimes c^n + 2(q-1)^n \mu_2 c^n \otimes 1 \\
&+ 2(q-1)^n \mu_2^2 (q^2-1)^n (g_2^n - 1) g_1^n \otimes a_1^n \\
&+ (q^2-1)^n(q-1)^n \mu_2^2 a_1^n \otimes 1 + (q^2-1)^n (q-1)^n \mu_2^2 g_1^n \otimes a_1^n \\
=&(g_1g_2^2)^n \otimes [d^n + 2(q-1)^n \mu_2 c^n + (q^2-1)^n (q-1)^n \mu_2^2 a_1^n] \\
&+ [d^n + 2(q-1)^n \mu_2 c^n + (q^2-1)^n(q-1)^n \mu_2^2 a_1^n] \otimes 1 \\
&- 2(q-1)^n \mu_2 g_1^n g_2^n \otimes c^n + (q^2-1)^n (q-1)^n \mu_2^2 (-2g_2^n + 1) g_1^n \otimes a_1^n \\
&+ 2(q-1)^n \mu_2 (g_1g_2)^n \otimes c^n + 2(q-1)^n (q^2-1)^n \mu_2^2 (g_1^n)(g_2^n - 1) \otimes a_1^n \\
&+ (q^2-1)^n (q-1)^n \mu_2^2 g_1^n \otimes a_1^n \\
=&(g_1g_2^2)^n \otimes \omega + \omega \otimes 1.
\end{aligned}$$

Thus ω is $((g_1g_2^2)^n, 1)$-primitive and so, by Lemma 2.1 (3), we have

$$\omega = d^n + 2(q-1)^n \mu_2 c^n + (q^2-1)^n (q-1)^n \mu_2^2 a_1^n = \gamma((g_1g_2^2)^n - 1) \tag{14}$$

where $\gamma = 0$ if $(\chi_1\chi_2^2)^n \neq \epsilon$ or $(g_1g_2^2)^n = 1$.

Remark 2.4: If $b_{21}^n \neq 1$, then $\gamma = 0$. Indeed, $(\chi_1\chi_2^2)^n(g_2) = b_{21}^n \neq 1$, so $(\chi_1\chi_2^2)^n \neq \epsilon$, forcing $\gamma = 0$.

We find a characterization for the liftings of Nichols algebras of type B_2 similar to that in [6] for type A_2. The following lemma from [6] will be useful.

LEMMA 2.5 ([6, Lemma 3.4 (i)]): *Let X, Y, Z be elements in a k-algebra, α, β scalars in k and n a natural number. If $YX = \alpha XY + Z$ and $ZX = \beta XZ$, then*

$$YX^n = \alpha^n X^n Y + (\sum_{i=0}^{n-1} (\alpha^i \beta^{n-1-i}) X^{n-1} Z$$

and, if $\alpha \neq \beta$ and $\alpha^n = \beta^n$, then for $n > 1$,

$$YX^n = \alpha^n X^n Y. \qquad \blacksquare$$

Define a Hopf algebra U^+ in the category ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ by $U^+ := k < x_1, x_2, z, u|\mathcal{N} >$, where $\mathcal{N}$ is the set of the first six relations defined at the beginning of this section, namely:

$$z = x_2x_1 - b_{21}x_1x_2, \tag{15}$$
$$u = x_2z - b_{21}b_{22}zx_2, \tag{16}$$
$$x_1z = b_{12}zx_1, \tag{17}$$
$$x_2u = b_{21}b_{22}^2ux_2, \tag{18}$$
$$uz = b_{11}b_{21}zu, \tag{10}$$
$$x_1u = b_{21}^{-1}b_{12}ux_1 + b_{21}^{-1}(b_{22}^{-1} - 1)z^2, \tag{20}$$

and the comultiplication, action and coaction are defined such that x_1, x_2 are primitive and $h \cdot x_i = \chi_i(h)x_i, \delta(x_i) = g_i \otimes x_i$. To see that U^+ is well defined, we note that if we make the free algebra F generated by x_1 and x_2 into a braided Hopf algebra with x_1 and x_2 primitives, then $\Delta_F(\mathcal{N}) \subseteq \mathcal{N} \otimes F + F \otimes \mathcal{N}$, and this induces a braided Hopf algebra structure on U^+. U^+ has a PBW basis $\{x_2^iu^jz^rx_1^s|\ i,j,r,s \geq 0\}$. This follows from the fact that U^+ can be constructed from $k[x_1]$ by adjoining z, u, x_2 by iterated Ore extensions defined by the relations (15)–(20).

We define U to be the Radford biproduct $U^+\#k\Gamma$.

THEOREM 2.6: *Let $\mu_1, \mu_2 \in \{0, 1\}$ and $\lambda, \gamma \in k$ such that*

$$\mu_i = 0 \quad \text{if } g_i^n = 1 \text{ or } \chi_i^n \neq \epsilon; \tag{21}$$
$$\lambda = 0 \quad \text{if } g_1^ng_2^n = 1 \text{ or } \chi_1^n\chi_2^n \neq \epsilon; \tag{22}$$
$$\gamma = 0 \quad \text{if } (\chi_1\chi_2^2)^n \neq \epsilon \text{ or } (g_1g_2^2)^n = 1. \tag{23}$$

Then the two-sided ideal J of U generated by the elements

$$y_i := x_i^n - \mu_i(g_i^n - 1), \quad i = 1, 2;$$
$$v := z^n + \mu_2(q-1)^nx_1^n - \lambda(g_1^ng_2^n - 1);$$
$$w := u^n + 2(q-1)^n\mu_2z^n + (q^2-1)^n(q-1)^n\mu_2^2x_1^n - \gamma((g_1g_2^2)^n - 1),$$

is a Hopf ideal of U. Moreover, $A = A(\Gamma, V, (\mu_i)_i, \lambda, \gamma) = U/J$ is a pointed Hopf algebra of dimension $n^4|\Gamma|$ with coradical $k\Gamma$, and $gr(A) \simeq \mathcal{B}(V)\#k\Gamma$, where V is our fixed Yetter-Drinfeld module of type B_2.

Proof: Since, by the arguments preceding (9), (11) and (14), J is generated by skew-primitive elements, J is a Hopf ideal. Next we verify some commutation relations. We have

$$\begin{aligned}
x_2x_1^n &= b_{21}^n x_1^n x_2 \text{ by Lemma 2.5 with } Y = x_2, X = x_1, Z = z, \alpha = b_{21}, \beta = b_{21}b_{22}^2;\\
zx_1^n &= b_{21}^n x_1^n z \text{ by (17)};\\
ux_1^n &= (x_2z - b_{21}b_{22}zx_2)x_1^n = (b_{21}^2)^n x_1^n u;\\
zx_2^n &= b_{12}^n x_2^n z \text{ by Lemma 2.5 with } Y = z, X = x_2, Z = u, \alpha = b_{12}b_{22}, \beta = b_{12};\\
ux_2^n &= b_{12}^n x_2^n u \text{ by (18)};\\
x_1z^n &= b_{12}^n z^n x_1 \text{ by (17)};\\
x_2z^n &= b_{21}^n z^n x_2 \text{ by Lemma 2.5 with } Y = x_2, X = z, Z = u, \alpha = b_{21}b_{22},\\
&\qquad \beta = b_{11}b_{21};\\
uz^n &= b_{21}^n z^n u \text{ by (19)};\\
x_1u^n &= b_{12}^{2n} u^n x_1 \text{ by Lemma 2.5 with } Y = x_1, X = u, Z = z^2, \alpha = b_{12}^2b_{22}^2, \beta = b_{12}^2;\\
x_2u^n &= b_{21}^n u^n x_2 \text{ by (18)};\\
zu^n &= b_{12}^n u^n z \text{ by (19)}.
\end{aligned}$$

It remains to find the commutation between x_1 and x_2^n.

Now $x_1x_2^n = b_{12}b_{22}^2(x_2x_1 - z)x_2^{n-1}$, and

$$zx_2^t = (b_{12}b_{22})^t x_2^t z - b_{12}^t\left(\sum_{i=0}^{t-1} b_{22}^i\right)b_{22}x_2^{t-1}u$$

by Lemma 2.5.

Thus

$$\begin{aligned}
x_1x_2^n = {} & b_{12}b_{22}^2x_2x_1x_2^{n-1} - (b_{12}b_{22}^2)(b_{12}b_{22})^{n-1}x_2^{n-1}z\\
& + (b_{12}b_{22}^2)b_{12}^{n-1}\left(\sum_{i=0}^{n-2} b_{22}^i\right)b_{22}x_2^{n-2}u.
\end{aligned}$$

Then

$$x_1x_2^n = (b_{12}b_{22}^2)^n x_2^n x_1 + \alpha x_2^{n-1}z + \beta x_2^{n-2}u$$

and we show that $\alpha = \beta = 0$.

It is easy to see that

$$\alpha = -b_{12}^n \sum_{i=1}^{n} b_{22}^{2i}b_{22}^{n-i} = -b_{12}^n \sum_{i=1}^{n} b_{22}^i = 0,$$

and that

$$\begin{aligned}\beta =& b_{12}^n b_{22}^3 \left\{ \sum_{i=0}^{n-2} b_{22}^i + b_{22}^2 \sum_{i=0}^{n-3} b_{22}^i + b_{22}^4 \sum_{i=0}^{n-4} b_{22}^i + \cdots + b_{22}^{2(n-2)}(1) \right\} \\ =& b_{12}^n b_{22}^3 \{(q^{n-1}-1) + q^2(q^{n-2}-1) + \cdots + q^{2(n-2)}(q-1)\}/(q-1).\end{aligned}$$

Now the expression in brackets is just

$$\begin{aligned}&(q^{n-1} + q^2 q^{n-2} + q^4 q^{n-3} + \cdots + q^{2n-3}) - (1 + q^2 + q^4 + \cdots + q^{2(n-2)}) \\ =& q^{-1}(q^{n-1}-1)/(q-1) - (q^{2(n-1)}-1)/(q^2-1),\end{aligned}$$

and putting these expressions over a common denominator, we see that this is 0 and so $\beta = 0$.

We have proved that

$$x_1 x_2^n = b_{12}^n x_2^n x_1.$$

We show now that J is the right ideal generated by y_i, $i = 1, 2$; v; w.

Assume first that $b_{12}^n = 1$. Then for $h \in \Gamma$ and $i = 1, 2$, we have

$$hy_i = \chi_i^n(h) y_i h + (\chi_i^n(h) - 1)\mu_i (g_i^n - 1)h$$

and, by (21), we always have $(\chi_i^n(h) - 1)\mu_i = 0$. Also

$$x_2 y_1 = y_1 x_2 \quad \text{and} \quad x_1 y_1 = y_1 x_1.$$

Similar computation for the other generators shows that J is the right ideal as well as the two-sided ideal generated by y_1, y_2, v, w.

If $b_{12}^n \neq 1$, then by Remarks 2.2, 2.3 and 2.4, we must have $\mu_1 = \mu_2 = \lambda = \gamma = 0$ and then J is the two-sided ideal generated by x_1^n, x_2^n, z^n, u^n. The commutation relations show immediately that J is the right ideal generated by these elements.

We prove now that no non-zero linear combination of the elements $gx_2^i u^j z^r x_1^s$, with $g \in \Gamma$, $0 \le i, j, r, s \le n-1$, lies in J. This will imply that the dimension of $A = U/J$ is n^4 and also that $J \cap k\Gamma = 0$, so $k\Gamma$ embeds in A. To show this we proceed as in the proof of [8, Proposition 1.10]. Assume that

$$\sum_{g,i,j,r,s} \alpha_{g,i,j,r,s}\, g x_2^i u^j z^r x_1^s = \sum_{i=1,2} y_i f_i + v f_3 + w f_4$$

for some $f_1, f_2, f_3, f_4 \in U$ and some scalars $\alpha_{g,i,j,r,s}$, not all equal to zero. The commutation relations show that U is a free module with basis

$$\{x_2^i u^j z^r x_1^s \mid 0 \le i, j, r, s \le n-1\}$$

over the subalgebra B of U generated by Γ and x_1^n, x_2^n, z^n, u^n. If we write f_i, $1 \leq i \leq 4$ in terms of this basis, we see that there exist some $F_1, F_2, F_3, F_4 \in B$ such that

$$\sum_{i=1,2} y_i F_i + vF_3 + wF_4 \in k\Gamma - \{0\}.$$

Clearly B is isomorphic as an algebra to an Ore extension R obtained from $k\Gamma$ by adjoining the indeterminates Y_2, Y_3, Y_4, Y_1 (identified with x_2^n, u^n, z^n, x_1^n) in that order via Ore extensions with zero derivations. This shows that the relation

$$\sum_{i=1,2} (Y_i - \mu_i(g_i^n - 1))q_i + (Y_4 + \mu_2(q-1)^n Y_1 - \lambda(g_1^n g_2^n - 1))q_3$$
$$+(Y_3 + 2(q-1)^n \mu_2 Y_4 + (q^2-1)^n(q-1)^n \mu_2^2 Y_1 - \gamma((g_1 g_2^2)^n - 1))q_4 \in k\Gamma - \{0\}$$

holds in R for some $q_i, 1 \leq i \leq 4$. The universal property for Ore extensions (see [8, Lemma 1.1]) shows that there exists an algebra morphism θ: $R \to k\Gamma$ acting as identity on Γ and such that $\theta(Y_i) = \mu_i(g_i^n - 1)$ for $1 \leq i \leq 2$, $\theta(Y_4) = -\mu_1\mu_2(q-1)^n(g_1^n - 1) + \lambda(g_1^n g_2^n - 1)$, and

$$\begin{aligned}\theta(Y_3) = &-2(q-1)^n\mu_2(-\mu_1\mu_2(q-1)^n(g_1^n - 1) + \lambda(g_1^n g_2^n - 1)) \\ &-(q^2-1)^n(q-1)^n\mu_1\mu_2^2(g_1^n - 1) + \gamma((g_1g_2^2)^n - 1).\end{aligned}$$

Then applying θ to the above equation we obtain that $0 \in k\Gamma - \{0\}$, a contradiction.

We have thus proved that the dimension of A is $n^4|\Gamma|$ and that $k\Gamma$ embeds in A. Since A is generated by Γ and the skew-primitive elements x_1, x_2, we see that A is pointed and the coradical of A is $k\Gamma$.

For the last claim, we consider the algebra morphism ϕ: $U \to gr(A)$ which takes x_i to the image of x_i modulo $k\Gamma$ in the homogeneous component of degree 1 of $gr(A)$. Since A is generated as an algebra by Γ and x_1, x_2, this algebra morphism is surjective. On the other hand, since $x_i^n \in k\Gamma$ in A, we have that $x_i^n = 0$ in $gr(A)$. Similarly, regarding the images of z, respectively u, in $gr(A)$, they have degrees 2, respectively 3, and we also get that $z^n = 0$ and $u^n = 0$ in $gr(A)$. Therefore ϕ induces a surjective algebra morphism ψ from $U/(x_1^n, x_2^n, z^n, u^n) \simeq \mathcal{B}(V)\#k\Gamma$ to $gr(A)$, and this morphism must be an isomorphism because of the dimensions. Obviously ψ is also a coalgebra morphism, and this ends the proof. ∎

THEOREM 2.7: *Let A be a pointed Hopf algebra with coradical $k\Gamma$ and such that $gr(A) \simeq \mathcal{B}(V)\#k\Gamma$, where V is our fixed Yetter–Drinfeld module of type B_2, such* *that n is odd and $n \neq 5$. Then $A \simeq A(\Gamma, V, (\mu_i)_i, \lambda, \gamma)$ for some $g_i, \chi_i, \mu_i, \lambda, \gamma$ as in Theorem 2.6.*

Proof: Suppose first that $n \neq 3$. We have shown that relations (4), (6), (7), (8) hold in A, so there exists a Hopf algebra morphism ϕ: $A(\Gamma, V, (\mu_i)_i, \lambda, \gamma) \to A$ which takes x_i to a_i for $i = 1, 2$. By [4, Lemma 2.2] we have that A is generated as an algebra by Γ, a_1 and a_2, so ϕ is surjective. The dimension implies now that ϕ is an isomorphism.

Now suppose that $n = 3$; in this case relation (6) has not been verified. Since $n \neq 5$, by the proof of Lemma 2.1 (5), $\chi_1\chi_2^3 \neq \epsilon$, so that $da_2 - b_{12}a_2d \in P_{g_1g_2^3}^{\chi_1\chi_2^3}$ means $g_1g_2^3 = g_i$ and $\chi_1\chi_2^3 = \chi_i$ for i=1 or 2. If $i = 2$, then the argument is the same as in Lemma 2.1. If $i = 1$, then $g_2^3 = 1$ and $\chi_2^3 = \epsilon$. But then $a_2^3 = 0$ by (9) and thus $a_2^3 \rightharpoonup a_1 = 0 = a_2 \rightharpoonup d = a_2d - b_{21}b_{22}^2da_2$. Relation (6) follows from (1). ∎

3. Quasi-isomorphism of liftings

Recall that Hopf algebras A and B are quasi-isomorphic if one is a cocycle twist of the other and this implies that their categories of comodules are monoidally Morita–Takeuchi equivalent (see [12] or [17]). If one of A or B is pointed or finite dimensional, then the converse holds. If A and B are quasi-isomorphic, we write $A \sim B$.

As well as the infinite families of nonisomorphic Hopf algebras of the same dimension obtained by lifting quantum linear spaces that were mentioned in the introduction, such infinite families can also be easily constructed from liftings of $\mathcal{B}(V)\#k\Gamma$, where $V \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ is of type A_2 or B_2. Recall that V of type A_2 means that $V = kx_1 \oplus kx_2$ and there exist $g_1, g_2 \in \Gamma$ and $\chi_1, \chi_2 \in \hat{\Gamma}$ such that for all $g \in \Gamma$ and $i = 1, 2$, we have

$$g \to x_i = \chi_i(g)x_i \quad \text{and} \quad \delta(x_i) = g_i \otimes x_i.$$

Also for $b_{ij} = \chi_j(g_i)$ as in Section 2,

$$(24) \qquad b_{12}b_{21}b_{11} = 1 = b_{21}b_{12}b_{22}, \quad b_{11} = b_{22} = q, \quad \text{where } q \text{ is a primitive } n\text{th root of unity,}$$

so that the Cartan matrix A determined from the braiding matrix $B = (b_{ij})$ is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The liftings of $\mathcal{B}(V)\#k\Gamma$ where $V \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ is of type A_2 were determined in [6] where $n > 3$ or Γ is cyclic of order 3.

For x_1, x_2 as above, a Hopf algebra $U^+ \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ is defined in [6, Definition 3.5] by

$$U^+ = k < x_1, z, x_2 | z = x_1x_2 - b_{12}x_2x_1,\ zx_1 = b_{21}x_1z,\ x_2z = b_{21}zx_2 >,$$

where the x_i are primitive elements. Then U is defined to be the Radford biproduct $U^+\#k\Gamma$.

THEOREM 3.1 ([6, Theorems 3.6 and 3.7]): *Let A be a lifting of $\mathcal{B}(V)\#k\Gamma$ for V of type A_2 as described above. If n is an odd integer greater than 3 or if Γ is cyclic of order 3, then $A \cong U/J$, where J is the Hopf ideal of U generated by the skew-primitives*

$$x_i^n - \mu_i(g_i^n - 1) \quad \text{where } \mu_i \in \{0,1\} \text{ and } \mu_i = 0 \text{ if } g_i^n = 1 \text{ or } \chi_i^n \neq \epsilon,$$
$$z^n + \mu_1(q-1)^n x_2^n - \lambda(g_1^n g_2^n - 1) \quad \text{where } \lambda = 0 \text{ if } g_1^n g_2^n = 1 \text{ or } \chi_1^n\chi_2^n \neq \epsilon.$$

Example 3.2 (cf. [6, Section 3]): Let $\Gamma =< g >$ be cyclic of order 49. Let q be a primitive 7-th root of unity. Let $\chi \in \hat{\Gamma}$ be defined by $\chi(g) = q$. Define $g_1 = g, g_2 = g^4, \chi_1 = \chi, \chi_2 = \chi^2$. Let $V = kx_1 \oplus kx_2$ where $x_i \in V_{g_i}^{\chi_i}$. Then the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} q & q^2 \\ q^4 & q^8 = q \end{bmatrix}$$

and the Cartan matrix A is $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ so V is of type A_2.

Let $A(\lambda)$ be the Hopf algebra

$$U/ < x_1^7 - (1-g^7), x_2^7 - (1-(g^4)^7), z^7 - (q-1)^7(1-(g_4)^7) - \lambda(1-(gg^4)^7) >$$

as in Theorem 3.1.

For $\lambda \neq \omega$, $A(\lambda) \not\cong A(\omega)$. For suppose $f\colon A(\lambda) \to A(\omega)$ is a Hopf algebra isomorphism. Then $f(g) = g$ since, if $f(g) = g^4$, then $f(g^4) = g^{16} \neq g$. Thus $f(x_1) = \alpha y_1 + \delta(1-g^7)$, where y_i is the counterpart of x_i in $A(\omega)$. Commutation with g shows that $\delta = 0$ and $\alpha^7 = 1$. Similarly, $f(x_2) = \beta y_2$ where $\beta^7 = 1$. Then $f(z) = \alpha\beta(y_1y_2 - b_{12}y_2y_1) = \alpha\beta\omega$ and

$$(\alpha\beta)^7\omega^7 = \omega^7 = (q-1)^7(1-g^{28}) - \lambda(1-g^{35}) = (q-1)^7(1-g^{28}) - \omega(1-g^{35}),$$

so $\lambda = \omega$. ∎

We now describe the liftings for the remaining case $n = 3$. Let $\mathcal{U}^+$ be the free algebra in the indeterminates x_1 and x_2. This is a Hopf algebra in the category

${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ by taking the x_i's to be primitive elements where $\delta(x_i) = g_i \otimes x_i$ and $h \rightharpoonup x_i = \chi_i(h)x_i$ for all $h \in \Gamma$. Denote $z = x_1x_2 - qx_2x_1$. We define $\mathcal{U}$ as the Radford biproduct $\mathcal{U}^+\#k\Gamma$.

PROPOSITION 3.3: *Suppose $V \in {}^{k\Gamma}_{k\Gamma}\mathcal{YD}$ is of type A_2 and $n = 3$. Then any lifting of $\mathcal{B}(V)\#k\Gamma$ is isomorphic to $\mathcal{U}/J$ where $\mathcal{U}$ is as defined above and J is the Hopf ideal of $\mathcal{U}$ generated by:*

$$x_i^3 - \mu_i(g_i^3 - 1) \quad \text{where } \mu_i \in \{0,1\} \text{ and } \mu_i = 0 \text{ if } g_i^3 = 1 \text{ or } \chi_i^3 \neq \epsilon;$$
$$zx_i - q^i x_i z - \gamma_i(g_i^2 g_j - 1), \quad i \in \{1,2\}, i \neq j, \gamma_i = 0 \text{ if } g_i^2 g_j = 1 \text{ or } \chi_i^2\chi_j \neq \epsilon;$$
$$z^3 + \mu_1(q-1)^3 x_2^3 + (1-q)\gamma_1(zx_2 - q^2 x_2 z) - \lambda(g_1^3 g_2^3 - 1)$$
$$\text{where } \lambda = 0 \text{ if } g_1^3 g_2^3 = 1 \text{ or } \chi_1^3\chi_2^3 \neq \epsilon.$$

Proof: Let A be a lifting of $\mathcal{B}(V)\#k\Gamma$. Let $a_i \in A$ be the lifting of $x_i \in V$ as in the B_2 case and as in [6], $a_i^3 = \mu_i(g_i^3 - 1)$ where $\mu_i \in \{0,1\}$ and $\mu_i = 0$ if $g_i^3 = 1$ or $\chi_i^3 \neq \epsilon$. For $c = a_1a_2 - b_{12}a_2a_1$ as in [6], then it is shown in [3, Lemma 3.1] that $ca_1 - b_{21}a_1c \in P^{\chi_1^2\chi_2}_{g_1^2g_2}$ and $a_2c - b_{21}ca_2 \in P^{\chi_1\chi_2^2}_{g_1g_2^2}$. If $\chi_i^2\chi_j \neq \epsilon$ for $i,j \in \{1,2\}$, or if Γ is cyclic of order 3, then we are in the situation of Theorem 3.1. But for $n = 3$ and Γ not cyclic of order 3, we could have $\chi_i^2\chi_j = \epsilon$; then the matrix $B = (b_{ij}) = \begin{pmatrix} q & q \\ q & q \end{pmatrix}$ with $q^3 = 1$. From now on, we assume this is the case. Then

$$ca_i - q^i a_i c = \gamma_i(g_i^2 g_j - 1) \tag{25}$$
for some $\gamma_i \in k$ with $\gamma_i = 0$ if $g_i^2 g_j = 1$ or $\chi_i^2\chi_j \neq \epsilon$.

Now as in the calculation of (3), we have

$$\triangle(c) = g_1g_2 \otimes c + c \otimes 1 + (1-q^2)a_1g_2 \otimes a_2 = X + Y + (1-q^2)Z.$$

Then we have the following commutation relations:

$$\begin{aligned}
XY &= qYX;\\
XZ &= qZX + \gamma_2 q^2 T \quad \text{where } T = a_1g_1g_2^2 \otimes (g_1g_2^2 - 1);\\
YZ &= q^2 ZY + \gamma_1 S \quad \text{where } S = (g_1^2 g_2 - 1)g_2 \otimes a_2;\\
XT &= q^2 TX;\\
ZT &= qTZ;\\
YT &= qTY + \gamma_1 W \quad \text{where } W = (g_1^2g_2 - 1)g_1g_2^2 \otimes (g_1g_2^2 - 1);\\
XS &= q^2 SX + \gamma_2 W;
\end{aligned}$$

$$\begin{aligned} YS =& qSY; \\ ZS =& q^2SZ; \\ TS =& q^2ST. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} \triangle(c^3) =&(X+Y+(1-q^2)Z)^3 \\ =&(g_1g_2)^3 \otimes c^3 + c^3 \otimes 1 + (q-1)^3 a_1^3 g_2^3 \otimes a_2^3 \\ &+ (1-q)\gamma_1\gamma_2 g_1 g_2^2 (g_1^2 g_2 - 1) \otimes (g_1 g_2^2 - 1). \end{aligned}$$

Let $v = c^3 + \mu_1(q-1)^3 a_2^3 + (1-q)\gamma_1\gamma_2(g_1g_2^2-1)$. Then

$$\begin{aligned} \triangle(v) =&(g_1g_2)^3 \otimes c^3 + c^3 \otimes 1 + (q-1)^3\mu_1(g_1^3-1)g_2^3 \otimes a_2^3 \\ &+ (1-q)\gamma_1\gamma_2 g_1 g_2^2(g_1^2g_2-1) \otimes (g_1g_2^2-1) + \mu_1(q-1)^3 a_2^3 \otimes 1 \\ &+ \mu_1(q-1)^3 g_2^3 \otimes a_2^3 + (1-q)\gamma_1\gamma_2(g_1g_2^2-1) \otimes 1 \\ &+ (1-q)\gamma_1\gamma_2 g_1 g_2^2 \otimes (g_1g_2^2-1) \\ =&(g_1g_2)^3 \otimes [c^3 + (q-1)^3\mu_1 a_2^3 + (1-q)\gamma_1\gamma_2(g_1g_2^2-1)] \\ &+ [c^3 + \mu_1(q-1)^3 a_2^3 + (1-q)\gamma_1\gamma_2(g_1g_2^2-1)] \otimes 1 \\ &- \mu_1(q-1)^3 g_2^3 \otimes a_2^3 - (1-q)\gamma_1\gamma_2 g_1g_2^2 \otimes (g_1g_2^2-1) \\ &+ \mu_1(q-1)^3 g_2^3 \otimes a_2^3 + (1-q)\gamma_1\gamma_2 g_1g_2^2 \otimes (g_1g_2^2-1) \\ =&(g_1g_2)^3 \otimes v + v \otimes 1, \end{aligned}$$

and thus $v \in P_{g_1^3g_2^3}^{\chi_1^3\chi_2^3}$. If $\chi_1^3\chi_2^3 \neq \epsilon$, then $\chi_1^3\chi_2^3 = \chi_i$ for $i = 1, 2$, yielding $q = 1$, which is a contradiction. Therefore, $v = \lambda(g_1^3g_2^3 - 1)$ for some $\lambda \in k$. Now an argument similar to the one in Theorem 2.6 shows that the elements $ha_1^ic^ja_2^k$, $h \in \Gamma, 0 \leq i, j, l, k \leq 2$ are a basis for A, and the same argument as in Theorem 2.7 completes the proof. ∎

Let $A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2)$ denote the Hopf algebra $\mathcal{U}/J$, where $\mathcal{U}$ is the Hopf algebra defined just before Proposition 3.3 and J is the Hopf ideal generated by the skew primitives:

$$\begin{gathered} x_i^n - \mu_i(g_i^n - 1) \quad \text{where } i \in \{1,2\}, \mu_i \in \{0,1\} \text{ and } \mu_i = 0 \text{ if } g_i^n = 1 \text{ or } \chi_i^n \neq \epsilon; \\ zx_1 - b_{21}x_1z - \gamma_1(g_1^2g_2 - 1), \quad \gamma_1 = 0 \text{ if } g_1^2g_2 = 1 \text{ or } \chi_1^2\chi_2 \neq \epsilon; \\ zx_2 - b_{21}^{-1}x_2z - \gamma_2(g_2^2g_1 - 1), \quad \gamma_2 = 0 \text{ if } g_2^2g_1 = 1 \text{ or } \chi_2^2\chi_1 \neq \epsilon; \\ z^n + \mu_1(q-1)^n x_2^n + (1-q)\gamma_1(zx_2 - b_{21}^{-1}x_2z) - \lambda(g_1^ng_2^n - 1) \\ \text{where } \lambda = 0 \text{ if } g_1^ng_2^n = 1 \text{ or } \chi_1^n\chi_2^n \neq \epsilon. \end{gathered}$$

Then by Theorem 3.1 and Proposition 3.3, all liftings of Nichols algebras of type A_2 are of this form. Now we show that all $A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2)$ with the same γ_i are quasi-isomorphic. We write $H \sim H'$ if the Hopf algebras H and H' are quasi-isomorphic. We use the key theorem from [12] together with comments from [13].

Recall that for K a Hopf algebra, the set $\mathrm{Alg}(K, k)$ is a group under the convolution product with the inverse to $\psi \in \mathrm{Alg}(K, k)$ being given by $\psi \circ S$ where S is the antipode of K. The left action of $\mathrm{Alg}(K, k)$ on K is given by $\psi x = (\mathrm{Id}_K \otimes \psi)\Delta(x)$ and the right action by $x\psi = (\psi \otimes \mathrm{Id}_K)\Delta(x)$. Two Hopf ideals I, J in a Hopf algebra K are said to be conjugate if there is an algebra map ψ from K to k such that $J = \psi I \psi^{-1}$. Also, if K is a subHopf algebra of a Hopf algebra H and J is a Hopf ideal of K, then (J) will denote the Hopf ideal in H generated by J.

THEOREM 3.4 ([12, Theorem 2], [13]): *Suppose that K is a Hopf subalgebra of a Hopf algebra H. Let I, J be Hopf ideals of K. If there is an algebra map ψ from K to k such that $J = \psi I \psi^{-1}$ and $H/(\psi I)$ is nonzero, then $H/(\psi I)$ is an $(H/(I), H/(J))$-biGalois object and so the quotient Hopf algebras $H/(I), H/(J)$ by the Hopf ideals $(I), (J)$ in H generated by I, J are monoidally Morita–Takeuchi equivalent.*

In the application of Masuoka's theorem, the following lemma will be useful.

LEMMA 3.5: *Let K be a Hopf algebra containing $(g_i, 1)$-primitives $x_i, i = 1, \ldots, t$. Let J be the Hopf ideal of K generated by the x_i and let L be the Hopf ideal generated by $x_i - \lambda_i(g_i - 1), i = 1, \ldots, t$. Let ψ be an algebra map from K to k such that $\psi(x_i) = \lambda_i$ and $\psi(h) = 1$ for h grouplike. Then J and L are conjugate ideals in K.*

Proof: Since $S(x_i) = -g_i^{-1}x_i, \psi(S(x_i)) = -\lambda_i$. Thus

$$\psi^{-1}x_i\psi = (\psi \otimes Id \otimes \psi^{-1})(\Delta^2 x_i) = \psi(g_i)g_i(-\lambda_i) + \psi(g_i)x_i + \psi(x_i) = x_i - \lambda_i(g_i - 1),$$

and $\psi^{-1}J\psi = L$. ∎

THEOREM 3.6: *For V of type A_2, and $A = A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2)$ a lifting of $\mathcal{B}(V)\#k\Gamma$, then A is quasi-isomorphic to any other lifting*

$$A(\Gamma, V, \mu_1', \mu_2', \lambda', \gamma_1, \gamma_2).$$

If $\gamma_1 = 0$ then $A \sim A(\Gamma, V, \mu_1', \mu_2', \lambda', 0, \gamma_2')$ and if $\gamma_2 = 0$ then

$$A \sim A(\Gamma, V, \mu_1', \mu_2', \lambda', \gamma_1', 0).$$

In particular, if $n > 3$ or Γ is cyclic of order 3, then all liftings of $\mathcal{B}(V)\#k\Gamma$ are quasi-isomorphic.

Proof: First note that if $b_{ij}^n \neq 1$, for $i \neq j$, then for $n > 3$ by an argument similar to Remark 2.2, $b_{ji}^n \neq 1$ and the only possible lifting is the trivial one, $A(\Gamma, V, 0, 0, 0, 0, 0)$. For $n = 3$ this follows from the proof of Proposition 3.3.

Now assume $b_{21}^n = b_{12}^n = 1$. Since n is odd, then also $b_{21}^{n(n-1)/2} = 1$. First, we show that for a given $\gamma_1, \gamma_2, \mu_1$, $A(\Gamma, V, \mu_1, 0, 0, \gamma_1, \gamma_2)$ and

$$A = A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2)$$

are quasi-isomorphic for any $\mu_2 \in \{0, 1\}$ and any λ.

Let $M_{\mu_1} = \mathcal{U}/ < x_1^n - \mu_1(g_1^n - 1),\ zx_1 - b_{21}x_1z - \gamma_1(g_1^2g_2 - 1),\ zx_2 - b_{12}b_{22}x_2z - \gamma_2(g_1g_2^2 - 1) >$ where $\mathcal{U}$ is the Hopf algebra defined just before Proposition 3.3. Note that $v = z^n + \mu_1(q-1)^n x_2^n + (1-q)\gamma_1\gamma_2(g_1g_2^2 - 1)$ is $(g_1^ng_2^n, 1)$-primitive in M_{μ_1}. If $\gamma_1 = \gamma_2 = 0$, this follows from the proof of Theorem 3.1 [6, Theorem 3.6], and if some $\gamma_i \neq 0$, then we are in the situation of Proposition 3.3. We note that $A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2) = M_{\mu_1}/ < x_2^n - \mu_2(g_2^n - 1), v - \lambda(g_1^ng_2^n - 1) >$.

Since M_{μ_1} is obtained by adjoining x_1, z and x_2 via Ore extensions to $k\Gamma$ and then factoring by a Hopf ideal, then we may let K be the Hopf subalgebra of M_{μ_1} generated by Γ', the subgroup of Γ generated by g_1 and g_2, and by x_2^n and z^n, i.e., by g_1, g_2, x_2^n and v. Since $b_{ij}^n = 1$, the g_i commute with x_2^n and z^n. Also z^n and x_2^n commute. For, if $\gamma_2 = 0$, then $x_2z = b_{21}zx_2$ and, since $b_{21}^n = 1$, the commutation is clear. If $\gamma_2 \neq 0$, then $n = 3$, and the commutation of x_2^3 and z^3 follows from Lemma 2.5 with $Y = z, X = x_2, \alpha = q^2, Z = \gamma_2(g_1g_2^2 - 1), \beta = 1$. Thus the Hopf algebra K is a commutative polynomial algebra over $k\Gamma'$ in the indeterminates x_2^n and z^n.

Now let $\psi\colon K \to k$ be the algebra map defined by $\psi(g_1) = \psi(g_2) = 1$, $\psi(x_2^n) = \mu_2$ and $\psi(v) = \lambda$. Then $\psi^{-1}(x_2^n) = -\mu_2$ and $\psi^{-1}(v) = -\lambda$. By Lemma 3.5, the ideal J generated by the skew-primitives x_2^n and v and the ideal I generated by the skew-primitives $x_2^n - \mu_2(g_2^n - 1)$ and $v - \lambda(g_1^ng_2^n - 1)$ are conjugate in K. Also $(\psi J) \neq M_{\mu_1}$ since (ψJ) is the Hopf ideal generated by $x_2^n + \mu_2 g_2^n$ and $v + \lambda g_1^n g_2^n$. Thus $M_{\mu_1}/(J) = A(\Gamma, V, \mu_1, 0, 0, \gamma_1, \gamma_2)$ and $M_{\mu_1}/(I) = A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2)$ are quasi-isomorphic.

Next, let

$$M = \mathcal{U}/ < x_2^n,\ zx_1 - b_{21}x_1z - \gamma_1(g_1^2g_2 - 1),\ zx_2 - b_{12}b_{22}x_2z - \gamma_2(g_1g_2^2 - 1) > .$$

Then $M/ < x_1^n, v > \cong A(\Gamma, V, 0, 0, 0, \gamma_1, \gamma_2)$ and $M/ < x_1^n - (g_1^n - 1), v > \cong A(\Gamma, V, 1, 0, 0, \gamma_1, \gamma_2)$ and showing that $J = < x_1^n, v >$ and $I = < x_1^n - (g_1^n - 1), v >$

are conjugate Hopf ideals in some Hopf subalgebra of M will complete the proof. Let K be the Hopf subalgebra of M generated by g_1, g_2, x_1^n and z^n. Again, g_1 and g_2 commute with x_1^n and z^n and if $\gamma_1 = 0$, then $zx_1 = b_{21}x_1z$ in M, so that x_1^n and z^n commute. If $\gamma_1 \neq 0$, then the commutation of x_1^3 and z^3 again follows from Lemma 2.5. Define an algebra map φ: $K \to k$ by $\varphi(g_1) = \varphi(g_2) = 1$ for $h \in \Gamma$, $\varphi(x_1^n) = 1$, $\varphi(v) = 0$. Then Lemma 3.5 again yields that J and I are conjugate in K and so $A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma_1, \gamma_2) \sim A(\Gamma, V, 0, 0, 0, \gamma_1, \gamma_2)$. If $n > 3$ or Γ is cyclic of order 3, then $\gamma_1 = \gamma_2 = 0$, and all liftings of $\mathcal{B}(V)\#k\Gamma$ are quasi-isomorphic.

Assume now that $n = 3$ and that we are in the situation of Proposition 3.3 with $\chi_1^2\chi_2 = \chi_1\chi_2^2 = \epsilon$. Let $L = \mathcal{U}/ < x_1^3, x_2^3, zx_1 - qx_1z, z^3 >$, and let K be the commutative subHopf algebra of L generated by g_1, g_2 and the skew-primitive $zx_2 - q^2x_2z$. Define an algebra map φ: $K \to k$ by $\varphi(g_1) = \varphi(g_2) = 1$ and $\varphi(zx_2 - q^2x_2z) = \gamma_2$. Then as above, the Hopf ideals J generated by $zx_2 - q^2x_2z$ and I generated by $zx_2 - q^2x_2z - \gamma_2(g_1g_2^2 - 1)$ are conjugate in S and so $L/(J)$ and $L/(I)$ are quasi-isomorphic, i.e., $A(\Gamma, V, 0, 0, 0, 0, 0) \sim A(\Gamma, V, 0, 0, 0, 0, \gamma_2)$. Similarly, $A(\Gamma, V, 0, 0, 0, 0, 0) \sim A(\Gamma, V, 0, 0, 0, \gamma_1, 0)$. ∎

QUESTION: *For $n = 3$ and γ_1, γ_2 nonzero, is*

$$A(\Gamma, V, 0, 0, 0, \gamma_1, \gamma_2) \sim A(\Gamma, V, 0, 0, 0, 0, 0)?$$

Added in proof: A. Masuoka has answered this question in the affirmative. His method of proof is very much in the style of the proofs in [12].

Now we consider the case where V is of type B_2 and $n \neq 5$. If $A \cong U/J$ is the lifting determined by the scalars $\mu_1, \mu_2, \lambda, \gamma$ as in Theorem 2.6, then we write $A = A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma)$.

THEOREM 3.7: *For $V \in {}_{k\Gamma}^{k\Gamma}\mathcal{YD}$ of type B_2 and $n \neq 5$, any two liftings of $\mathcal{B}(V)\#k\Gamma$ are quasi-isomorphic.*

Proof: As in the proof of Theorem 3.6, if $b_{ij}^n \neq 1$, then only the lifting $A(\Gamma, V, 0, 0, 0, 0)$ is possible. Therefore we assume that $b_{12}^n = b_{21}^n = 1$.

We first show that $A(\Gamma, V, 0, \mu_2, 0, 0) \sim A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma)$ for fixed μ_2 and any μ_1, λ, γ. Let $M(\mu_2) = U/ < x_2^n - \mu_2(g_2^n - 1) >$, where U is defined just before Theorem 2.6.

Recall that v, ω defined in Section 2, equations (10), (13), are skew-primitives, as is x_1^n. We write x_1^n, v, ω also for the images of these elements in $M(\mu_2)$ and note that they are still skew-primitive.

Now let K be the Hopf subalgebra of $M(\mu_2)$ generated by by g_1, g_2, x_1^n, z^n and u^n. As in the proof of Theorem 3.6, as an algebra K is a commutative polynomial algebra over $K\Gamma'$ where Γ' is generated by g_1, g_2.

Let ψ be the k-linear map from K to k defined by $\psi(g_1) = \psi(g_2) = 1$, $\psi(x_1^n) = \mu_1$, $\psi(v) = \lambda$, $\psi(\omega) = \gamma$. Then ψ defines an algebra map from K to k. In K, let J be the Hopf ideal generated by x_1^n, v and ω and let L be the Hopf ideal generated by the skew-primitives $x_1^n - \mu_1(g_1^n - 1), v - \lambda(g_1^n g_2^n - 1)$ and $\omega - \gamma(g_1^n g_2^{2n} - 1)$. Then by Lemma 3.5, J and L are conjugate in K and, by [12],

$$A(\Gamma, V, 0, \mu_2, 0, 0) \cong M(\mu_2)/(J) \sim M(\mu_2)/(L) \cong A(\Gamma, V, \mu_1, \mu_2, \lambda, \gamma).$$

Finally, we show that $A(\Gamma, V, 0, 0, 0, 0) \sim A(\Gamma, V, 0, 1, 0, 0)$. Let

$$M = U/ < x_1^n >,$$

let K be the commutative Hopf subalgebra of M generated by g_1, g_2, x_2^n, z^n and u^n, let ψ: $K \to k$ be the algebra map defined by $\psi(g_i) = 1 = \psi(x_2^n), \psi(v) = \psi(\omega) = 0$. Now the same argument finishes the proof. ∎

4. A generalization of the q-binomial theorem

BY

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Throughout, we work over a field k, not necessarily algebraically closed.

From Theorem 1.1 (i), it is straightforward to prove that

(26)
$$\binom{n+1}{i}_q \binom{i}{j}_q = \binom{n}{i-1}_q \binom{i-1}{j-1}_q + q^i \binom{n}{i}_q \binom{i}{j}_q + q^j \binom{n}{i-1}_q \binom{i-1}{j}_q.$$

Now we prove the generalized quantum binomial theorem used in the calculations in this paper. However, this theorem is interesting in its own right.

THEOREM A.1: *Suppose that $q \in k^*$ and $\lambda \in k$ and, for x, t, z in some k-algebra, we have the following relations:*

$$xz = qzx; \quad zt = qtz; \quad xt = qtx + \lambda z^2. \tag{27}$$

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Then

$$(x+bz+t)^n = \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}_q \binom{i}{j}_q \nu(i-j)t^j z^{i-j} x^{n-i}$$

where $\nu = \nu_{b,\lambda}$ *is a function from* $\mathbf{N}$ *to* k *defined recursively by* $\nu(0) = 1, \nu(1) = b$, *and* $\nu(n) = b\nu(n-1) + \lambda(n-1)_q\nu(n-2)$, *for* $n \geq 2$.

Proof: The proof is by induction. The formula can easily be checked for $n = 1, 2$. Now assume that the formula holds for $n = k$, and we show that it is valid for $n = k+1$. First we note that it follows directly from Lemma 2.5 or from a simple induction argument that

$$xt^n = q^n t^n x + \lambda q^{n-1}(n)_q t^{n-1} z^2. \tag{28}$$

Then we compute

$$\begin{aligned}(x+bz+t)^{k+1} =& (x+bz+t)(x+bz+t)^k \\ =& \sum_{i=0}^{k}\sum_{j=0}^{i} \binom{k}{i}_q \binom{i}{j}_q \nu(i-j)(x+bz+t)t^j z^{i-j} x^{k-i} \\ =& \sum_{i=0}^{k}\sum_{j=0}^{i} \binom{k}{i}_q \binom{i}{j}_q \nu(i-j)((q^j t^j x + \lambda q^{j-1}(j)_q t^{j-1} z^2) z^{i-j} x^{k-i} \\ &+ q^j b t^j z^{i-j+1} x^{k-i} + t^{j+1} z^{i-j} x^{k-i}) \quad \text{by (28)} \\ =& \sum_{i=0}^{k}\sum_{j=0}^{i} \binom{k}{i}_q \binom{i}{j}_q \nu(i-j)(\lambda q^{j-1}(j)_q t^{j-1} z^{2+i-j} x^{k-i} \\ &+ q^i t^j z^{i-j} x^{k+1-i} + q^j b t^j z^{i-j+1} x^{k-i} + t^{j+1} z^{i-j} x^{k-i}).\end{aligned}$$

Now let $i' = i+1, j' = j-1, j'' = j+1$ and then

$$\begin{aligned}(x+bz+t)^{k+1} =& \sum_{i'=1}^{k+1}\sum_{j'=-1}^{i'-2} \binom{k}{i'-1}_q \binom{i'-1}{j'+1}_q \nu(i'-j'-2)\lambda q^{j'}(j'+1)_q \\ &\times t^{j'} z^{i'-j'} x^{k+1-i'} \\ &+ \sum_{i=0}^{k}\sum_{j=0}^{i} \binom{k}{i}_q \binom{i}{j}_q \nu(i-j) q^i t^j z^{i-j} x^{k+1-i} \\ &+ \sum_{i'=1}^{k+1}\sum_{j=0}^{i'-1} \binom{k}{i'-1}_q \binom{i'-1}{j}_q \nu(i'-j-1) q^j b t^j z^{i'-j} x^{k+1-i'} \\ &+ \sum_{i'=1}^{k+1}\sum_{j''=1}^{i'} \binom{k}{i'-1}_q \binom{i'-1}{j''-1}_q \nu(i'-j'') t^{j''} z^{i'-j''} x^{k+1-i'},\end{aligned}$$

and then, using the fact that $\binom{m}{s}_q = 0$ if $s > m$ or $s < 0$ and $(0)_q = 0$, we have that

$$
\begin{aligned}
&(x+bz+t)^{k+1}\\
=&\sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k}{i-1}_q\binom{i-1}{j}_q\frac{(i-j-1)_q}{(j+1)_q}\nu(i-j-2)\lambda q^j(j+1)_q t^j z^{i-j}x^{k+1-i}\\
&+\sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k}{i}_q\binom{i}{j}_q\nu(i-j)q^i t^j z^{i-j}x^{k+1-i}\\
&+\sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k}{i-1}_q\binom{i-1}{j}_q\nu(i-j-1)q^j b t^j z^{i-j}x^{k+1-i}\\
&+\sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k}{i-1}_q\binom{i-1}{j-1}_q\nu(i-j)t^j z^{i-j}x^{k+1-i}\\
=&\sum_{i=0}^{k+1}\sum_{j=0}^{i}(\binom{k}{i-1}_q\binom{i-1}{j}_q(b\nu(i-j-1)+\lambda\nu(i-j-2)(i-j-1)_q)q^j\\
&+\binom{k}{i}_q\binom{i}{j}_q\nu(i-j)q^i+\binom{k}{i-1}_q\binom{i-1}{j-1}_q\nu(i-j))t^j z^{i-j}x^{k+1-i}\\
=&\sum_{i=0}^{k+1}\sum_{j=0}^{i}\left(\binom{k}{i-1}_q\binom{i-1}{j-1}_q+q^i\binom{k}{i}_q\binom{i}{j}_q+q^j\binom{k}{i-1}_q\binom{i-1}{j}_q\right)\\
&\qquad\qquad\times\nu(i-j)t^j z^{i-j}x^{k+1-i}\\
&\text{by the definition of }\nu\\
=&\sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k+1}{i}_q\binom{i}{j}_q\nu(i-j)t^j z^{i-j}x^{k+1-i}\quad\text{by (26), as required.}\qquad\blacksquare
\end{aligned}
$$

Remarks A.2: (i) If $\lambda = 0$, then $(x+bz+t)^n = ((x+bz)+t)^n$ where $(x+bz)t = qt(x+bz)$, and so the same result may be obtained directly from the q-binomial theorem (Theorem 1.1(ii)).

(ii) Suppose q is a primitive nth root of unity. Then $\binom{n}{i}_q = 0$ unless $i = 0$ or $i = n$, and the formula in Theorem A.1 becomes $(x+bz+t)^n = x^n + \nu_{b,\lambda}(n)z^n + t^n$.

The description of $(x+bz+t)^n$ would now be complete if we had a general formula for $\nu_{b,\lambda}(s)$.

PROPOSITION A.3: *If $q = 1$ and $b \neq 0$ then, for $n \geq 0$,*

$$\nu(n) = \sum_{i=0}^{n/2} \binom{n}{2i} \frac{(2i)!}{2^i(i!)} b^{n-2i} \lambda^i.$$

Proof: Note that since $\binom{n}{2i} = 0$ if $2i > n$, the summation is from 0 to $\lfloor n/2 \rfloor$. First, if $n = 0$, we check that $\binom{0}{0}\frac{0!}{2^0 0!} b^0 \lambda^0 = 1$ and, if $n = 1$, the formula gives $\binom{1}{0}\frac{0!}{2^0 0!} b^1 = b$. Now suppose the formula holds for $n \leq k+1$ and we compute $\nu(k+2) = b\nu(k+1) + (k+1)\lambda\nu(k)$. By the induction assumption, this is

$$\begin{aligned}
&\sum_{i=0}^{(k+1)/2} \binom{k+1}{2i} \frac{(2i)!}{2^i(i!)} b^{k+2-2i} \lambda^i + \sum_{i=0}^{k/2} \binom{k}{2i} (k+1) \frac{(2i)!}{2^i(i!)} b^{k-2i} \lambda^{i+1} \\
&= \sum_{i=0}^{(k+2)/2} \binom{k+1}{2i} \frac{(2i)!}{2^i(i!)} b^{k+2-2i} \lambda^i \\
&\quad + \sum_{i'=0}^{(k+2)/2} \binom{k}{2i'-2} (k+1) \frac{(2i'-2)!}{2^{i'-1}((i'-1)!)} b^{k+2-2i'} \lambda^{i'} \\
&\quad \text{where } i' = i+1 \\
&= \sum_{i=0}^{(k+2)/2} (\binom{k+1}{2i} \frac{(2i)!}{2^i(i!)} + \binom{k+1}{2i-1} \frac{(2i-1)!}{2^{i-1}((i-1)!)}) b^{k+2-2i} \lambda^i \\
&= \sum_{i=0}^{(k+2)/2} \binom{k+2}{2i} \frac{(2i)!}{2^i i!} b^{n+2-2i} \lambda^i. \qquad \blacksquare
\end{aligned}$$

If $q \neq 1$, there is a formula for the computation of $\nu(n)$ in terms of α, β (α, β possibly lie in some extension field of k) where $\alpha + \beta = b$ and $\alpha\beta = \lambda/(q-1)$.

PROPOSITION A.4: *Suppose that $q \neq 1$. Let α, β be the roots of the polynomial $Y^2 - bY + \lambda/(q-1)$ in some extension field $\overline{k}$ of k. Then*

$$\nu_{b,\lambda}(n) = \sum_{i=0}^{n} \binom{n}{i}_q \beta^i \alpha^{n-i}.$$

Proof: Since $x + bz + t = (x + \alpha z) + (\beta z + t)$, and since $(x + \alpha z)(\beta z + t) = q(\beta z + t)(x + \alpha z)$, we have by the q-binomial theorem (Theorem 1.1) that

$$\begin{aligned}
(x + bz + t)^n &= \sum_{i=0}^{n} \binom{n}{i}_q (\beta z + t)^i (x + \alpha z)^{n-i} \\
&= \sum_{i=0}^{n} \binom{n}{i}_q (\sum_{j=0}^{i} \binom{i}{j}_q t^j (\beta z)^{i-j})(\sum_{k=0}^{n-i} \binom{n-i}{k}_q (\alpha z)^k x^{n-i-k}).
\end{aligned}$$

Comparing terms in this formula with the one in Theorem A.1, we see that

$$\sum_{m=0}^{n}\binom{n}{m}_q \nu(m)z^m x^{n-m} = \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}_q\binom{n-i}{k}_q(\beta z)^i(\alpha z)^k x^{n-i-k}$$
$$= \sum_{w=0}^{n}\sum_{i=0}^{w}\binom{n}{i}_q\binom{n-i}{w-i}_q \beta^i\alpha^{w-i}z^w x^{n-w}.$$

Comparing coefficients of $z^m x^{n-m}$, we obtain

$$\binom{n}{m}_q \nu(m) = \sum_{i=0}^{m}\binom{n}{i}_q\binom{n-i}{m-i}_q \beta^i\alpha^{m-i},$$

and then, letting $m=n$, we obtain the statement. ∎

COROLLARY A.5: *If q is a primitive nth root of unity, and $n>1$, then $\nu(n)=\alpha^n+\beta^n\in k$.*

If b=0, then $\nu(1)=b=0$, and since $\nu(2n+1)=b\nu(2n)+\lambda(2n)_q\nu(2n-1)$, it is clear that $\nu(2n+1)=0$ for $n\geq 0$. However, $\nu(0)=1, \nu(2)=\lambda(1)_q$, $\nu(4)=\lambda^2(3)_q$, and, in general, $\nu(2n)=\lambda^n(2n-1)_q(2n-3)_q\cdots(1)_q$.

COROLLARY A.6 (to Theorem A.1): *For x,z,t,q,λ satisfying (27),*

$$(x+t)^n = \sum_{i=0}^{n}\sum_{m=0}^{\lfloor \frac{i}{2}\rfloor}\binom{n}{i}_q\binom{i}{i-2m}_q \nu(2m)t^{i-2m}z^{2m}x^{n-i}.$$

Proof: Applying Theorem A.1 with $b=0$, we have

$$(x+t)^n = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}_q\binom{i}{j}_q \nu(i-j)t^j z^{i-j}x^{n-i}$$
$$= \sum_{i=0}^{n}\sum_{k=0}^{i}\binom{n}{i}_q\binom{i}{i-k}_q \nu(k)t^{i-k}z^k x^{n-i}$$
$$= \sum_{i=0}^{n}\sum_{m=0}^{\lfloor \frac{i}{2}\rfloor}\binom{n}{i}_q\binom{i}{i-2m}_q \nu(2m)t^{i-2m}z^{2m}x^{n-i}. \qquad ∎$$

Remark A.7: Suppose x,t,s are such that

$$xs=q^2sx,\ \ st=q^2ts \quad\text{and}\quad xt=qtx+\lambda s,$$

i.e., the relations (27) hold with $s=z^2$. Then if q is a primitive nth root of unity and n is odd, $(x+t)^n=x^n+t^n$.

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